

Lower bound on the value of the fine-structure constant

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Recently we have proposed the existence of a universal relation between the maximal electric charge and total mass of any weakly self-gravitating object: $Z \leq Z^* = \alpha^{-1/3} A^{2/3}$, where Z is the number of protons, A is the total baryon (mass) number, and $\alpha = e^2/\hbar c$ is the fine-structure constant. Motivated by this novel bound, we explore the (Z, A) -relation of atomic nuclei as deduced from the Weizsäcker semi-empirical mass formula. It is shown that *all* nuclei, including the meta-stable maximally charged ones, conform to the upper bound. Moreover, we suggest that the new charge-mass bound places an interesting constraint on the value of the fine-structure constant: $\alpha \gtrsim 1/323$.

The weak cosmic censorship conjecture (WCCC), introduced by Penrose forty years ago [1, 2], is one of the corner stones of general relativity. This principle asserts that spacetime singularities that arise in gravitational collapse are always hidden inside of black holes. The elimination of a black-hole horizon is ruled out by this hypothesis because that would expose naked singularities to distant observers.

Arguing from the cosmic censorship principle, we have proposed [3] the existence of a universal bound on the charge q of any weakly self-gravitating object of total energy μ : $q \leq \mu^{2/3} E_c^{-1/3}$, where E_c is the critical electric field for pair-production [4]. For charged objects with nuclear matter density the upper bound corresponds to

$$Z \leq Z^* = \alpha^{-1/3} A^{2/3}, \quad (1)$$

where Z and A are the number of protons and the total baryon number, respectively, and $\alpha \equiv e^2/\hbar c$ is the fine-structure constant. [We shall henceforth use natural units in which $c = 1$.]

This bound was inferred from the requirement that the WCCC be respected in a gedanken experiment in which a charged object falls into a charged black hole. The integrity of the black-hole horizon is respected provided Z is bounded as in Eq. (1). This relation limits the charges of objects such as atomic nuclei and quark nuggets [3, 5]. The intriguing feature of our derivation [3] is that it uses a principle whose very meaning stems from gravitation (the cosmic censorship principle) to derive a universal bound which has nothing to do with gravitation [written out fully, the bound (1) would involve \hbar and c , but not G]. This provides a striking illustration of the unity of physics.

It is of considerable interest to check the validity of the new charge-mass bound (1). For instance, Lead $^{208}_{82}\text{Pb}$, the largest known completely stable nucleus satisfies the relation $Z/A^{2/3} \simeq 2.33 < \alpha^{-1/3}$. Thus, this nucleus conforms to the upper bound (1). The largest known artificially made nucleus contains $Z = 118$ protons and a total number of $A = 294$ nucleons [6]. This nucleus

satisfies the relation $Z/A^{2/3} \simeq 2.67 < \alpha^{-1/3}$. Thus, one finds that this nucleus also respects the upper bound (1) [3].

Even heavier meta-stable nuclei are expected to be produced in the forthcoming years using accelerator production techniques. In fact, some calculations suggest that nuclei of $A \sim 300$ to 476 may have very long lifetimes [7]. Could these highly charged nuclei be able to threaten the validity of the cosmic censorship conjecture by violating the (Z, A) -bound (1)? In order to address this question, we shall investigate the charge-mass relation of atomic nuclei as deduced from the well-known semi-empirical mass formula [8–12].

Consider an atomic nucleus composed of Z protons and N neutrons. The total baryon number is given by $A = Z + N$. The charge and mass of a nucleus are given by

$$q = Z|e| \quad ; \quad \mu = Zm_p + Nm_n - \mathcal{E}_B \simeq Am_p, \quad (2)$$

where \mathcal{E}_B is the binding energy of the nucleus, which is typically much smaller than its mass. The binding energy \mathcal{E}_B of a nucleus (that is, the difference between its mass and the sum of the masses of its individual constituents) is well approximated by the semi-empirical mass formula, also known as Weizsäcker's formula [8–12]:

$$\mathcal{E}_B(A, Z) = a_V A - a_S A^{2/3} - a_C \frac{Z^2}{A^{1/3}} - a_A \frac{(A - 2Z)^2}{A}. \quad (3)$$

This well-known formula is based on the liquid drop model which treats the nucleus as a drop of incompressible nuclear fluid composed of protons and neutrons. The four terms on the r.h.s of Eq. (3) correspond to the cohesive binding of all the nucleons by the strong nuclear force, a surface energy term (which represents the fact that surface nucleons are less tightly bound as compared to bulk nucleons), the electrostatic mutual repulsion of the protons, and an asymmetry term (which represents

the fact that protons and neutrons occupy independent quantum momentum states) [8–12]. The coefficients in the semi-empirical mass formula are calculated by fitting to experimentally measured masses of nuclei [12]:

$$\begin{aligned} a_V &= 15.36(3) \quad ; \quad a_S = 16.42(8) \quad ; \\ a_C &= 0.691(2) \quad ; \quad a_A = 22.53(7) . \end{aligned} \quad (4)$$

To a rough approximation, the nucleus can be considered a sphere of uniform charge density. The potential energy of such a charge distribution is given by $\mathcal{E}_p = 3Q^2/5R$. It is well-known that the radii of atomic nuclei are well approximated by the empirical relation $R(A) = 1.219 \times A^{1/3}$ fm [12]. This is a direct consequence of the fact that the size of an individual nucleon is roughly given by its Compton length. Thus, one can write $R(A) \simeq A^{1/3} \times \frac{\xi \hbar}{m_p}$ fm, where m_p is the proton's mass and ξ is a constant of order unity (empirically one finds $\xi \simeq 5.788$). Substituting $R(A)$ and $Q = Z|e|$ into the expression of the potential energy \mathcal{E}_p , one finds

$$a_C \simeq \frac{3m_p \alpha}{5\xi} \equiv c\alpha , \quad (5)$$

where $c \equiv 3m_p/5\xi \simeq 97.38$. For $\alpha \simeq 1/137.036$ [14] one finds $a_C \simeq 0.711$. This estimated value of a_C is astonishingly close to the empirically measured one (less than 3% difference), see Eq. (4). Below we shall come back to this observation.

We shall first consider nuclei with the *largest* possible electric charge, $Z_{\max}(A)$, for a given value of the baryon number A . These nuclei pose the greatest challenge to the charge-mass bound (1). A nucleus may in principle be produced (and live even for a short duration of time) if it has a positive binding energy. The maximal values $Z_{\max}(A)$ are determined from Eq. (3) with the requirement $\mathcal{E}_B(A, Z_{\max}) \geq 0$ [with $\mathcal{E}_B(A, Z_{\max} + 1) < 0$]. It should be emphasized that, although such hypothetical nuclei are characterized by positive binding energy (that is, their masses are *smaller* than the sum of the masses of their individual constituents), they are expected to be short-lived. This is due to the fact that their binding energies are smaller than the corresponding binding energies of the stable nuclei, see Eq. (7) below. Thus, we do not expect to find such nuclei in nature. Nevertheless, such nuclei could in principle be produced artificially (and live for a short duration of time), and it is therefore of interest to study these nuclei in the context of the new charge-mass bound (1).

Noting that the second and fourth terms in the mass formula (3) are negative, one realizes that the binding energy $\mathcal{E}_B(A, Z)$ will become negative once the Coulomb energy overcomes the volume energy [13]. Thus, one may obtain a simple upper bound on the value of $Z_{\max}(A)$:

$$Z_{\max}(A) < \left(\frac{a_V}{a_C} \right)^{1/2} A^{2/3} . \quad (6)$$

Substituting the experimentally measured values of the coefficients a_V and a_C [12], one finds $Z_{\max}(A) < 4.71A^{2/3}$. We therefore obtain the series of inequalities $Z(A)/Z^*(A) \leq Z_{\max}(A)/Z^*(A) < 4.71/\alpha^{-1/3} < 0.91 < 1$. This implies that all atomic nuclei, including the meta-stable *maximally* charged ones (with the maximally allowed electric charge according to the semi-empirical mass formula. Most of these nuclei are yet to be produced artificially) conform to the new $Z - A$ bound (1).

In figure 1 we depict the actual ratio $Z_{\max}(A)/Z^*(A)$ as a function of the mass number A , where $Z_{\max}(A)$ is calculated from the full expression of the Weizsäcker semi-empirical mass formula (3). One indeed finds $Z_{\max}/Z^* < 1$ for *all* nuclei, with a maximal value of $\simeq 0.85$.

In figure 1 we also depict the ratio $Z_{\text{stable}}(A)/Z^*(A)$ as a function of the mass number A , where $Z_{\text{stable}}(A)$ is the number of protons of the most stable nucleus of mass number A . Here $Z_{\text{stable}}(A)$ is obtained by maximizing the binding energy $\mathcal{E}_B(A, Z)$ with respect to Z . This yields [3]

$$Z_{\text{stable}}(A) = \frac{A}{2} \frac{1}{1 + \frac{a_C}{4a_A} A^{2/3}} . \quad (7)$$

(For light nuclei this expression reduces to the canonical relation $Z = A/2$.) One finds that the ratio Z_{stable}/Z^* has a maximal value of $\simeq 0.56$ [3].

The dimensionless fine-structure constant α is one of the fundamental parameters of the standard model of particle physics. It has puzzled many scientists since its introduction by Sommerfeld [15] almost a century ago. The question of how to derive the numerical value of α from some underlying theory has been one of the most important open questions in modern physics [16–24]. As we shall now argue, one may deduce a bound on the value of the fine-structure constant α from the novel charge-mass bound (1). To that end, let us assume that α is a free parameter in Eqs. (1) and (5).

The requirement $Z_{\text{stable}}(A) \leq Z^*(A)$ yields the quadratic equation

$$\frac{a_C}{2a_A} A^{2/3} - \alpha^{1/3} A^{1/3} + 2 \geq 0 . \quad (8)$$

The inequality (8) would be satisfied for *all* A values provided the discriminant is non-positive [25]:

$$\alpha^{2/3} - \frac{4a_C}{a_A} \leq 0 . \quad (9)$$

Substituting $a_C = c\alpha$ from Eq. (5), one obtains the inequality:

$$\alpha \geq \left(\frac{a_A}{4c} \right)^3 . \quad (10)$$

Substituting the experimentally measured values of the coefficients a_A and c [12], one finds a lower bound on the value of the fine structure constant:

$$\alpha \gtrsim \frac{1}{5161.7}. \quad (11)$$

We shall now consider atomic nuclei with the largest possible electric charge for a given mass number A . In order to avoid a violation of the bound (1) (which would ultimately lead to a violation of the WCCC, see [3]), we must demand that any collection of nucleons which seems to violate (1) is actually unstable. More explicitly, we must demand that any collection of Z protons and $A - Z$ neutrons with $Z > \alpha^{-1/3} A^{2/3}$ is characterized by a negative binding energy. Substituting $Z^* = \alpha^{-1/3} A^{2/3}$ from the bound (1) into (3) and demanding that $\mathcal{E}_B(A, Z^* + 1) < 0$, one obtains the quadratic equation

$$(a_V - \frac{a_C}{\alpha^{2/3}} - a_A)A^{2/3} + (\frac{4a_A}{\alpha^{1/3}} - a_S)A^{1/3} - \frac{4a_A}{\alpha^{2/3}} \leq 0. \quad (12)$$

This inequality would be respected by *all* nuclei (by all A values) provided the discriminant is non-positive [26]:

$$(\frac{4a_A}{\alpha^{1/3}} - a_S)^2 + \frac{16a_A}{\alpha^{2/3}}(a_V - \frac{a_C}{\alpha^{2/3}} - a_A) \leq 0. \quad (13)$$

Substituting $a_C = c\alpha$ from Eq. (5), the inequality (13) can be written as a quadratic equation for $\alpha^{1/3}$:

$$a_S^2 \alpha^{2/3} - 8a_A(2c + a_S)\alpha^{1/3} + 16a_A a_V \leq 0. \quad (14)$$

This yields the lower bound

$$\alpha^{1/3} \geq \frac{4a_A(2c + a_S) - 4\sqrt{a_A^2(2c + a_S)^2 - a_S^2 a_A a_V}}{a_S^2}. \quad (15)$$

Substituting the experimentally measured values of the coefficients a_V , a_S , a_A , and c [12], one finds the lower bound [27]

$$\alpha \gtrsim \frac{1}{323.6}. \quad (16)$$

This bound is necessary for the validity of the charge-mass relation (1) [28]. Remarkably, the bound (16) on the value of the fine-structure constant is of the *same* order of magnitude as the experimentally measured value [14].

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- [25] Note that *if* α is identically zero, then condition (8) becomes a trivial one: $2 \geq 0$. Thus, our arguments cannot rule out this trivial solution.
- [26] Note that if $\alpha \equiv 0$, then condition (12) is satisfied trivially. Thus, our arguments cannot rule out this trivial solution.
- [27] The other solution of Eq. (13) yields the trivial inequality $\alpha^{1/3} \lesssim 140.9$.
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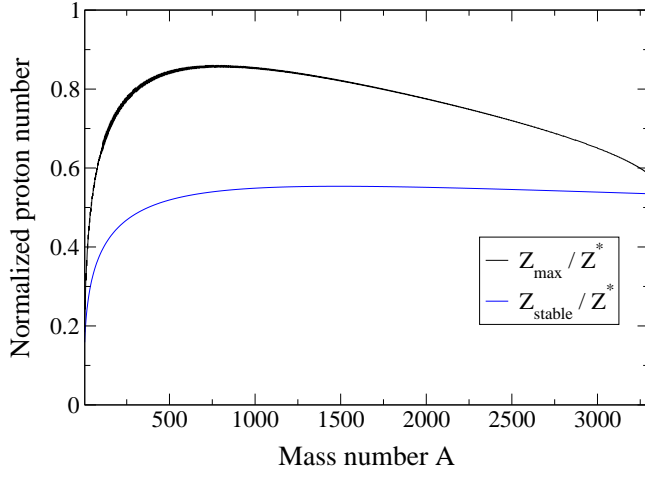


FIG. 1: The ratios $Z_{\max}(A)/Z^*(A)$ and $Z_{\text{stable}}(A)/Z^*(A)$ as a function of the mass number A . Here $Z_{\max}(A)$, $Z_{\text{stable}}(A)$, and $Z^*(A)$ are the maximally allowed number of protons in a nucleus of mass number A according to the Weizsäcker semi-empirical mass formula (3), the number of protons of the most stable nucleus of atomic mass A , and the maximally allowed number of protons for a given mass number A according to the novel upper bound (1), respectively. One realizes that these ratios are smaller than 1 for *all* nuclei (including hypothetical heavy nuclei with positive binding energy which are yet to be produced), with a maximal value of $\simeq 0.85$.